Control Variables for Finite Element Solution of Missile Trajectory Optimization

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Introduction

IN Ref. 1, a weak form of the necessary conditions for optimal control problems is presented. The resulting formulation is suitable for solution by the finite element method. This formulation is later extended[2] to treat problems with state control inequality constraints and applied[3] to trajectory optimization problems. One of the conclusions of Ref. 3 is that the method is suitable for real-time guidance. An algorithm for real-time guidance has now been developed[4] based on the computer code of Ref. 5.

In order to achieve a real-time algorithm, certain obstacles had to be overcome. Namely, in finite element solution of optimal control problems,[1–3] the Jacobian of the nonlinear algebraic equations must not become singular during the iterative solution of the nonlinear equations. Such a problem did develop in implementation of a formulation based on angle of attack and "bank" angle, thus motivating the development of a second formulation based on Lagrange's form of the equinoctial variables.[6] This Note describes both formulations and the change in the way the aerodynamic data were handled for implementation of the second formulation within the hybrid symbolic-numerical environment of Ref. 5.

Formulation of Problem

A missile, represented as a rigid body M, moving an inertial frame E with velocity V, is subject to inertial, aerodynamic, and body forces. For simplicity, and without loss of generality for the purpose of this development, E is assumed to be a flat and nonrotating Earth. The inertial properties of M are idealized to be those of a particle M^* of mass m. The gravity force at M^* is $mg\boldsymbol{k}$, where $\boldsymbol{k}$ is a unit vector directed vertically downward in E. The geometry of M is axisymmetric, and applied forces acting at M^* can be characterized in terms of a dextral triad of mutually perpendicular unit vectors $\boldsymbol{b}_i$ $(i = 1, 2, 3)$ fixed in a frame B. The unit vector $\boldsymbol{b}_1$ is fixed in M, parallel to its forward longitudinal axis. Thus, the applied forces consist of an axial force $(T - A)\boldsymbol{b}_1$ and a normal force $N\boldsymbol{n}$, where $\boldsymbol{n}$ is normal to $\boldsymbol{b}_1$ and lies in a plane that is parallel to both $\boldsymbol{b}_1$ and $\boldsymbol{V}$. The quantities A and N are known functions of Mach number, altitude, $v = |\boldsymbol{V}|$, and α, the angle between $\boldsymbol{V}$ and $\boldsymbol{b}_1$; T is a known function of time. In the degenerate case for which $\boldsymbol{V}$ is parallel to $\boldsymbol{b}_1$, $N = 0$. Initial conditions affect only the initial position and velocity of M^* in E. Note that the functionality of A and N with respect to Mach number and altitude is of no significance in the context of this Note.

Kinematics

First, we define a point O and a dextral triad of mutually perpendicular unit vectors $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ fixed in E. Letting $\boldsymbol{p}$ denote the position vector of M^* relative to O and $x_i = \boldsymbol{p} \cdot \boldsymbol{e}_i$, we can form the velocity of M^* in E as

$$\boldsymbol{V} = \dot{x}_1\boldsymbol{e}_1 + \dot{x}_2\boldsymbol{e}_2 + \dot{x}_3\boldsymbol{e}_3 \tag{1}$$

Denoting the unit vector tangent to the path of M^* as $\boldsymbol{w}_1$, we can also write the velocity as

$$\boldsymbol{V} = v\boldsymbol{w}_1 \tag{2}$$

where $\boldsymbol{w}_1$ is one member of a dextral triad of mutually perpendicular unit vectors $\boldsymbol{w}_i$ $(i = 1, 2, 3)$ fixed in a frame W. The unit vectors $\boldsymbol{w}_i$ have direction cosines relative to $\boldsymbol{e}_j$ denoted by

$$C_{ij} = \boldsymbol{w}_i \cdot \boldsymbol{e}_j \tag{3}$$

which form a matrix $[C]$. We will express $[C]$ in terms of Rodrigues parameters using the form found in Ref. 7, so that

$$[C] = \frac{\left(1 - \frac{1}{4}\{\theta\}^T\{\theta\}\right)[I] + \frac{1}{2}\{\theta\}\{\theta\}^T - [\tilde{\theta}]}{1 + \frac{1}{4}\{\theta\}^T\{\theta\}} \tag{4}$$

where $[C][C]^T = [C]^T[C] = [I]$, the 3×3 identity matrix, and

$$\{\theta\} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}, \qquad [\tilde{\theta}] = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix} \tag{5}$$

The $\tilde{(\,)}$ operator transforms any 3×1 column matrix such as $\{\theta\}$ into an antisymmetric 3×3 matrix of the form of $[\tilde{\theta}]$. Since there are three θ_i and only two parameters are needed to define the direction of $\boldsymbol{V}$, we need to constrain θ_i. We choose to do so with a nonholonomic constraint on the angular velocity below. Rodrigues parameters provide a singularity-free description of the orientation of W in E for rotations less than 180 deg. As will be seen below, there are additional benefits. Minimization of the number of state variables is thus achieved while retaining a pragmatic generality.

Equating expressions for $\boldsymbol{V}$ from Eqs. (1) and (2) and using Eq. (3), we obtain three kinematic equations of the form

$$\dot{x}_i = v\boldsymbol{w}_1 \cdot \boldsymbol{e}_i \tag{6}$$

or, in matrix form,

$$\{\dot{x}\} = v[C]^T\{e_1\} \tag{7}$$

where

$$\{e_1\} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}, \qquad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} \tag{8}$$

Denoting the angular velocity of W in E by $\boldsymbol{\omega}$, we let

$$\{\omega\} = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} \tag{9}$$

where $\omega_i = \boldsymbol{\omega} \cdot \boldsymbol{w}_i$. Based on Eq. (4), one can show that

$$\{\omega\} = \frac{1}{1 + \frac{1}{4}\{\theta\}^T\{\theta\}}\left[[I] - \tfrac{1}{2}[\tilde{\theta}]\right]\{\dot{\theta}\} \tag{10}$$

the inverse of which is given by

$$\{\dot{\theta}\} = \left[[I] + \tfrac{1}{2}[\tilde{\theta}] + \tfrac{1}{4}\{\theta\}\{\theta\}^T\right]\{\omega\} \tag{11}$$

We need the acceleration of M^*, which can be written as the inertial time derivative of $v\boldsymbol{w}_1$ so that

$$\begin{aligned} \boldsymbol{a} &= \dot{v}\boldsymbol{w}_1 + v\boldsymbol{\omega} \times \boldsymbol{w}_1 \\ &= \dot{v}\boldsymbol{w}_1 + v(\boldsymbol{w}_2\omega_3 - \boldsymbol{w}_3\omega_2) \end{aligned} \tag{12}$$

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In order for W to be well defined, we set $\omega_1 = 0$, which nonholonomically constrains the set of coordinates θ_i. In this way the orientation of frame W is governed only by the path of M^*. Now, only the orientation of $\boldsymbol{b}_1$ and the normal force need to be defined relative to W in order to define the aerodynamic forces. This we postpone until after we write the equations of motion in a general form.

Equations of Motion

For a missile of mass m, the equations of motion can be written in accordance with Newton's second law,

$$\boldsymbol{F} = m\boldsymbol{a} \tag{13}$$

where the force resultant $\boldsymbol{F}$ includes the normal force $N\boldsymbol{b}_2$, the axial force $A\boldsymbol{b}_1$, and the thrust $T\boldsymbol{b}_1$. Assuming $\boldsymbol{e}_3$ to be aligned with $\boldsymbol{k}$, the gravitational force is then $mg\boldsymbol{e}_3$. Thus,

$$\boldsymbol{F} = (T - A)\boldsymbol{b}_1 + N\boldsymbol{b}_2 + mg\boldsymbol{e}_3 \tag{14}$$

Introducing the column matrix of force components in the W system,

$$\{F\} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \tag{15}$$

where $F_i = \boldsymbol{F} \cdot \boldsymbol{w}_i$ and taking the dot product of Eq. (13) with $\boldsymbol{w}_i$, one obtains three equations of motion of the form

$$m\dot{v} = F_1, \qquad mv\omega_3 = F_2, \qquad -mv\omega_2 = F_3 \tag{16}$$

Equations (16b) and (16c) and the constraint $\omega_1 = 0$ can be written as

$$\{\omega\} = \frac{1}{mv}[\tilde{e_1}]\{F\} \tag{17}$$

The system is now governed by seven first-order ordinary differential equations. Equation (16a) is one scalar equation. The three equations embodied in Eq. (7) are already in a matrix form. Finally, Eqs. (11) and (17) can be combined into one matrix equation. Thus, the system equations are

$$\begin{aligned} \dot{v} &= \frac{1}{m}\{e_1\}^T\{F\} \\ \{\dot{x}\} &= v[C]^T\{e_1\} \\ \{\dot{\theta}\} &= \frac{1}{mv}\left[[I] + \tfrac{1}{2}[\tilde{\theta}] + \tfrac{1}{4}\{\theta\}\{\theta\}^T\right][\tilde{e_1}]\{F\} \end{aligned} \tag{18}$$

The initial values of x_i and v are known. Concerning θ_i, we know the initial velocity direction $\boldsymbol{w}_1$ so that $C_{11}(0)$, $C_{12}(0)$, and $C_{13}(0)$ are known. Thus, from Eq. (4) one can show

$$\begin{aligned} &[1 + C_{11}(0)]\theta_2(0) - \theta_1(0)C_{12}(0) + 2C_{13}(0) = 0 \\ &[1 + C_{11}(0)]\theta_3(0) - 2C_{12}(0) - \theta_1(0)C_{13}(0) = 0 \end{aligned} \tag{19}$$

For arbitrary values of $\theta_1(0)$, these equations can be solved for $\theta_2(0)$ and $\theta_3(0)$ if $C_{11}(0) \neq -1$.

A set of equations of motion for M, which is explicit in terms of control variables, is desired for ultimate use in trajectory optimization. Two sets of control variable formulations are presented, one in terms of orientation angles and the other in terms of Lagrange's form of the equinoctial variables.[6]

Control Formulation Based on Orientation Angles

We need only two parameters to represent the orientation of B in W. One way to proceed is to let B be tentatively oriented so that $\mathbf{b}_i$ coincides with $\boldsymbol{w}_i$ for $i = 1, 2, 3$. Then, we rotate B about $\boldsymbol{b}_1$ by a "bank" angle ϕ and then about $\boldsymbol{b}_3$ by the angle of attack α. This brings $\boldsymbol{b}_1$ into alignment with the axis of M during flight, and the normal force is assumed to act along $\boldsymbol{b}_2$. Thus,

$$\begin{Bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{Bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \boldsymbol{w}_3 \end{Bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & \cos\phi\sin\alpha & \sin\phi\sin\alpha \\ -\sin\alpha & \cos\phi\cos\alpha & \sin\phi\cos\alpha \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \boldsymbol{w}_3 \end{Bmatrix} \tag{20}$$

Now, by virtue of Eqs. (20) and (14), the force components may be written as

$$\begin{aligned} F_1 &= mgC_{13} + (T - A)\cos\alpha - N\sin\alpha \\ F_2 &= mgC_{23} + (T - A)\cos\phi\sin\alpha + N\cos\phi\cos\alpha \\ F_3 &= mgC_{33} + (T - A)\sin\phi\sin\alpha + N\sin\phi\cos\alpha \end{aligned} \tag{21}$$

and substituted directly into Eqs. (18).

A potential problem with this formulation is that ϕ can assume any value whatsoever when $\alpha = 0$. This affects the iterative solution procedure by causing the Jacobian to become singular in the vicinity of zero α, in turn causing the iteration to stop prematurely. The singularity is explicitly in the Hessian matrix of the Hamiltonian, which becomes indefinite in the vicinity of $\alpha = 0$. Such a problem developed in implementation of this formulation, thus motivating the development of another.

Control Formulation Based on Lagrangian Equinoctial Variables

This formulation overcomes the problem associated with the first one when α is near zero. To avoid the situation created by the vanishing of α in which ϕ then becomes undefined, we introduce two variables, β_2 and β_3, which resemble Lagrange's form of the equinoctial variables,[6] such that

$$\beta_2 = \cos\phi\tan\alpha, \qquad \beta_3 = \sin\phi\tan\alpha \tag{22}$$

so that

$$\beta^2 = \beta_2^2 + \beta_3^2 = \tan^2\alpha \tag{23}$$

Thus,

$$\begin{Bmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{1+\beta^2}} & \dfrac{\beta_2}{\sqrt{1+\beta^2}} & \dfrac{\beta_3}{\sqrt{1+\beta^2}} \\ \dfrac{-\beta}{\sqrt{1+\beta^2}} & \dfrac{\beta_2}{\beta\sqrt{1+\beta^2}} & \dfrac{\beta_3}{\beta\sqrt{1+\beta^2}} \\ 0 & \dfrac{-\beta_3}{\beta} & \dfrac{\beta_2}{\beta} \end{bmatrix} \begin{Bmatrix} \boldsymbol{w}_1 \\ \boldsymbol{w}_2 \\ \boldsymbol{w}_3 \end{Bmatrix} \tag{24}$$

and the measure numbers of the force vector can now be written as

$$\begin{aligned} F_1 &= mgC_{13} + \frac{T - A}{\sqrt{1+\beta^2}} - \frac{N\beta}{\sqrt{1+\beta^2}} \\ F_2 &= mgC_{23} + \frac{(T - A)\beta_2}{\sqrt{1+\beta^2}} + \frac{N\beta_2}{\beta\sqrt{1+\beta^2}} \\ F_3 &= mgC_{33} + \frac{(T - A)\beta_3}{\sqrt{1+\beta^2}} + \frac{N\beta_3}{\beta\sqrt{1+\beta^2}} \end{aligned} \tag{25}$$

This formulation has no kinematic singularity at $\alpha = 0$. However, before it can be used in a hybrid symbolic-numerical computational environment such as found in Refs. 5 and 8, apparent mathematical singularities need to be dealt with. Control variables β_2 and β_3, when combined with the $(\ )^*$ operator notation introduced below, provide a singularity-free formulation by which to determine the trajectory of M^*.

Operator Notation to Exhibit Absence of Singularities

Equations (25) need to be altered because N and A are functions of α whereas β appears explicitly in the coefficients of N and A, as well as that of T. To circumvent this, we can express all quantities in terms of β instead and group all the explicit β dependence into implicit functions that can be splined or approximated as a series. One way to do this is to introduce

$$A = \mu v^2 C_A, \qquad N = \mu v^2 C_N \tag{26}$$

where

$$\mu = \tfrac{1}{2}\rho S_A \tag{27}$$

Here, ρ is the density of air, S_A is the aerodynamic reference area of the missile, and C_N and C_A are considered functions of β, Mach number, and altitude. Note that characteristically C_N is an odd function of β, and C_A is an even one. Thus, the force components become

$$\begin{aligned}
F_1 &= mgC_{13} + \frac{T}{\sqrt{1+\beta^2}} - \frac{\mu v^2 C_A}{\sqrt{1+\beta^2}} - \frac{\mu v^2 C_N \beta}{\sqrt{1+\beta^2}}\\
F_2 &= mgC_{23} + \frac{T\beta_2}{\sqrt{1+\beta^2}} - \frac{\mu v^2 C_A \beta_2}{\sqrt{1+\beta^2}} + \frac{\mu v^2 C_N \beta_2}{\beta\sqrt{1+\beta^2}}\\
F_3 &= mgC_{33} + \frac{T\beta_3}{\sqrt{1+\beta^2}} - \frac{\mu v^2 C_A \beta_3}{\sqrt{1+\beta^2}} + \frac{\mu v^2 C_N \beta_3}{\beta\sqrt{1+\beta^2}}
\end{aligned} \tag{28}$$

Now, upon substitution of the even functions of β

$$\begin{aligned}
Q &= \frac{1}{\sqrt{1+\beta^2}}\\
C_a &= \frac{C_A}{\sqrt{1+\beta^2}}\\
C_n &= \frac{C_N}{\beta\sqrt{1+\beta^2}}
\end{aligned} \tag{29}$$

which can be conveniently represented as splines, Eqs. (28) simplify to

$$\begin{aligned}
F_1 &= mgC_{13} + TQ - \mu v^2\left(C_a + C_n\beta^2\right)\\
F_2 &= mgC_{23} + \left[TQ + \mu v^2 (C_n - C_a)\right]\beta_2\\
F_3 &= mgC_{33} + \left[TQ + \mu v^2 (C_n - C_a)\right]\beta_3
\end{aligned} \tag{30}$$

In construction of the nonlinear algebraic equations and the associated Jacobian, we will ultimately need to form first and second derivatives of the force components with respect to the control variables β_2 and β_3. Such partial derivatives of any function of β must be found from the chain rule. This involves derivatives of the function with respect to β times one of the following partial derivatives:

$$\frac{\partial\beta}{\partial\beta_2} = \frac{\beta_2}{\beta}, \qquad \frac{\partial\beta}{\partial\beta_3} = \frac{\beta_3}{\beta} \tag{31}$$

which has the effect of introducing indeterminate forms (0/0) in the vicinity of $\beta = 0$ every time there is differentiation with respect to one of the control variables.

The first derivative of any even function of β will be odd, but its first derivative divided by β will be even. Motivated by this observation, we introduce the notation $(\ )^* = (\ )'/\beta$ such that $(\ )' = \partial(\ )/\partial\beta$. Now,

$$Q^* = \frac{Q'}{\beta}, \qquad C_a^* = \frac{C_a'}{\beta}, \qquad C_n^* = \frac{C_n'}{\beta} \tag{32}$$

and so forth for higher derivatives:

$$\begin{aligned}
Q^{**} &= \frac{(Q^*)'}{\beta} = \frac{1}{\beta}\left(\frac{Q'}{\beta}\right)'\\
C_a^{**} &= \frac{(C_a^*)'}{\beta} = \frac{1}{\beta}\left(\frac{C_a'}{\beta}\right)'\\
C_n^{**} &= \frac{(C_n^*)'}{\beta} = \frac{1}{\beta}\left(\frac{C_n'}{\beta}\right)'
\end{aligned} \tag{33}$$

This way, if any of these functions, say Q, is represented as a power series,

$$Q = \sum_{i=0,2,4,\ldots}^{n} Q_i\beta^i \tag{34}$$

one can show that

$$\begin{aligned}
Q^* &= \sum_{i=2,4,6,\ldots}^{n} iQ_i\beta^{i-2}\\
Q^{**} &= \sum_{i=4,6,8,\ldots}^{n} i(i-2)Q_i\beta^{i-4}
\end{aligned} \tag{35}$$

Once these are calculated, we then can find

$$Q' = \beta Q^*, \qquad Q'' = \beta^2 Q^{**} + Q^* \tag{36}$$

We note that if the expansion of Q is limited to cubic order about the origin, say a cubic spline, then $Q^* = 2Q_2$ and $Q^{**} = 0$.

Of course, this operator does not change the results of finding derivatives. It just removes the myriad of indeterminate forms that would arise if the derivatives were carried out in a purely symbolic environment such as MACSYMA. Also, this operator greatly simplifies the expressions for the second derivatives. For example,

$$\begin{aligned}
\frac{\partial^2 F_1}{\partial\beta_3^2} &= T\left(Q^* + \beta_3^2 Q^{**}\right) - \mu v^2\left[C_a^* + 2C_n + C_n^*\beta^2\right.\\
&\quad \left.+ \beta_3^2\left(C_a^{**} + 4C_n^* + C_n^{**}\beta^2\right)\right]
\end{aligned} \tag{37}$$

$$\frac{\partial^2 F_3}{\partial\beta_2^2} = \left\{T\left(Q^* + \beta_2^2 Q^{**}\right) + \mu v^2\left[C_n^* - C_a^* + \beta_2^2 (C_n^{**} - C_a^{**})\right]\right\}\beta_3 \tag{38}$$

which, as can be seen, are well behaved in the limit of small β. It can be shown, in fact, that F_i and all their first and second derivatives with respect to β_2 and β_3 are as well. For $\beta = 0$ it can be shown that

$$\left.\frac{\partial^2 F_\gamma}{\partial\beta_\delta\,\partial\beta_\eta}\right|_{\beta=0} = 0 \qquad \delta, \eta, \gamma = 2, 3 \tag{39}$$

while

$$\left.\frac{\partial^2 F_1}{\partial\beta_2\,\partial\beta_3}\right|_{\beta=0} = 0 \tag{40}$$

$$\left.\frac{\partial^2 F_1}{\partial\beta_2^2}\right|_{\beta=0} = \left.\frac{\partial^2 F_1}{\partial\beta_3^2}\right|_{\beta=0} = -T - \mu v^2[C_a^*(0) + 2C_n^*(0)] \tag{41}$$

where the fact that $Q^*(0) = -1$ is used. Moreover, it can be easily shown that the Hessian matrix of the Hamiltonian for this formulation can never become indefinite in the vicinity of $\alpha = 0$.

Concluding Remarks

The first formulation based on orientation angles for control variables leads to nonconvergent behavior if, during the iterative solution procedure, α gets too close to zero; this is because of a singularity in the formulation at $\alpha = 0$. The second development, based on Lagrange's form of the equinoctial variables, solves this problem; however, without additional work the second formulation creates implementation problems in a hybrid symbolic-numerical computation environment. To overcome these difficulties, the $(\,)^*$ operator, developed herein, can be straightforwardly implemented in these codes.

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Minimal Time Change Detection Algorithm for Reconfigurable Flight Control Systems

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Introduction

THERE exists a significant interest in enhancing current flight systems to reconfigure the controls in order to maintain adequate levels of performance even if there are complete failures in one or more of the actuators or sensors. The problem of identifying aircraft system parameters from flight test data and the application of those parameters to flight control has been successfully solved for linear models when there are no severe changes in the aircraft and environment. Even when the control design incorporates a degree of robustness, system parameters may drift enough to degrade its performance below an acceptable level.

Adaptive controls are considered to be a promising approach for a possible solution. An adaptive control problem arises whenever the system parameters are unknown or are subject to unknown variation with usually a small change. Taking into account abrupt (or drastic) changes in statistical models, which include any faults in the system, appears as a natural complement to the adaptive techniques. It means that the detection of the abrupt change is essential in the design of a control system for these kinds of problem. A new performance index proposed in this paper is the speed of adaptation—how quickly the system could determine when a change occurs.

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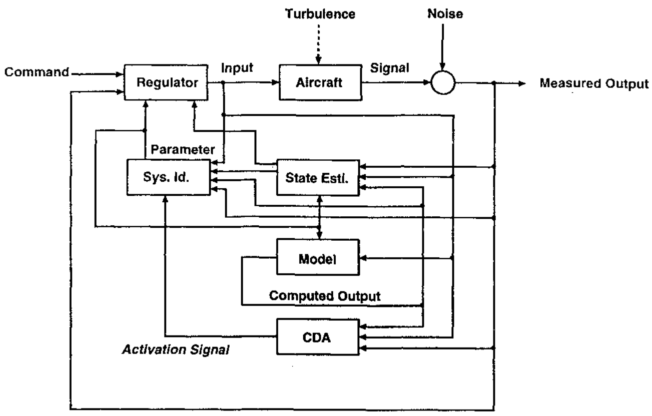

Fig. 1 Block diagram of control system for aircrafts with CDA.

The development of the change detection problem was stimulated by Wald[1] in 1947 when *Sequential Analysis* was published and a sequential probability ratio test was introduced. In 1954, a sequential process inspection scheme was proposed by Page[2] to detect a change in the mean by testing a weighted sum of the last few observations, i.e., a moving average. The most significant works in the theoretical properties of this rule are done by Shiryayev[3] and Lorden.[4]

A new, robust algorithm developed in this paper is called the minimal time change detection algorithm (MTCDA), which minimizes the time delay in detecting a change in a linear system for a fixed false-alarm probability. The block diagram shown in Fig. 1 explains the aircraft control system equipped with a change detection algorithm (CDA). Before the CDA detects a change, the regulator is designed based on the initial parameter values. But after the change is detected, the CDA block generates an activation signal to reinitiate system identification. The system identification process finds a new set of parameter values for the changed system. The regulator uses the new parameter values to build a new control law. Therefore, a reconfigurable control system can be designed that is suitable for a system that undergoes random changes. Algorithm complexity is definitely an issue in detecting each possible change and the time of each change.

Minimal Time Change Detection Algorithm

Detection of a Known Change

The following state-space form of a linear discrete-time stochastic model is considered:

$$x_{n+1} = Ax_n + [B_1 + (B_2 - B_1)\chi(n-\theta)]u_n \tag{1}$$

$$v_n = Cx_n + [D_1 + (D_2 - D_1)\chi(n-\theta)]u_n + GN_n \tag{2}$$

where θ is a change in time, $x_n \in R^n$, $u_n \in R^p$, $v_n \in R^m$, $GG^* > 0$, and $\chi(m)$ is 1 if $m \geq 0$ and 0 otherwise. The input is assumed sufficiently rich enough to detect a given change. Also only single change is considered, which means the changes are assumed to be detected one after the other. Based on this model, the basic problem to be solved is to detect a known change in control gain matrices as quickly as possible. This is appropriate for a system that operates in two alternate driving points and is restricted to the class of models with a change of parameters that is known. The probability of making a false detection also needs to be minimized since the signals are corrupted by a white Gaussian noise.

The problem in finding the detection time τ can be formulated as below by minimizing the average delay

$$E[(\tau - \theta)^+] \tag{3}$$

subject to a given false-alarm probability α, i.e.,

$$\Pr[\tau < \theta] \leq \alpha \tag{4}$$